

Representations of vertex operator algebras and bimodules

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Abstract

For a vertex operator algebra V , a V -module M and a nonnegative integer n , an $A_n(V)$ -bimodule $\mathbf{A}_n(M)$ is constructed and studied. The connection between $\mathbf{A}_n(M)$ and intertwining operators are discussed. Moreover, the $A_n(V)$ -bimodule $A_{t,s}(V)$ is a quotient of $\mathbf{A}_n(V)$ for all $s, t \leq n$. In the case that V is rational, $\mathbf{A}_n(M)$ for irreducible V -module M is given explicitly.

1 Introduction

The associative algebra $A_n(V)$ for any vertex operator algebra V and a nonnegative integer n was constructed in [DLM3] such that $A_{n-1}(V)$ is a quotient of $A_n(V)$ and there is a bijection between irreducible admissible V -modules and irreducible $A_n(V)$ -modules which cannot factor through $A_{n-1}(V)$. Moreover, V is rational if and only if $A_n(V)$ are finite dimensional semisimple associative algebras for all n . In the case $n = 0$, $A_0(V)$ is exactly the algebra $A(V)$ investigated in [Z]. An $A(V)$ -bimodule $A(M)$ for any V -module M was also introduced in [FZ] to deal with the intertwining operators and fusion rules.

Following the $A(M)$ -theory from [FZ] we construct a sequence of $A_n(V)$ -bimodules $\mathbf{A}_n(M)$ for any V -module M and a nonnegative integer n such that $\mathbf{A}_{n-1}(M)$ is a quotient of $\mathbf{A}_n(M)$ and $\mathbf{A}_0(M) = A(M)$. Moreover, the $A_n(V)$ -bimodule $A_{t,s}(V)$ for $s, t \leq n$ defined in [DJ] is a quotient of $\mathbf{A}_n(M)$. It is established that if V is rational then there is a linear isomorphism from the space $\mathcal{I}_{M^i M^j}^{M^k}$ of intertwining operators to

$$\mathrm{Hom}_{A_n(V)}(\mathbf{A}_n(M^i) \otimes_{A_n(V)} M^j(s), M^k(t))$$

for $s, t \leq n$ where $M^q = \bigoplus_{m \geq 0} M^q(m)$ ($q = i, j, k$) are the irreducible V -module such that $M^q(0) \neq 0$. This result is a generalization of that obtained in [FZ] when $n = 0$. The bimodule structure of $\mathbf{A}_n(M)$ for any irreducible module for rational V is given explicitly.

¹Supported by a NSF grant.

²Supported in part by China Postdoctor grant 2012M521688.

One can regard \mathbf{A}_n as a functor from the V -module category to the $A_n(V)$ -bimodule category. One important property of the functor $\mathbf{A}_0 = A$ is that A respects to the tensor product at least for rational vertex operator algebra. That is, $A(M \boxtimes N) = A(M) \otimes_{A(V)} A(N)$ for any V -modules M, N where $M \boxtimes N$ is the tensor product of V -modules as studied in [HL1, HL2, H]. Unfortunately, this is not true for general n . This can be seen clearly from the explicit $A_n(V)$ -bimodule structure of $\mathbf{A}_n(M)$.

Another interesting result about $\mathbf{A}_n(M)$ is the relation between $\mathbf{A}_n(M)$ and $\mathbf{A}_n(M')$ where M' is the contragredient module of M as defined in [FHL]. It is well known from the bimodule theory that $\mathbf{A}_n(M)^*$ is also an $A_n(V)$ -bimodule in an obvious way. We show that if V is rational and C_2 -cofinite then $\mathbf{A}_n(M)^*$ is isomorphic to $\mathbf{A}_n(M')$ for any irreducible V -module M . The proof involves a relation on the fusion matrices associated to M and M' [DJX]. This explains why we can only prove the isomorphism between $\mathbf{A}_n(M)^*$ and $\mathbf{A}_n(M')$ under rationality and C_2 -cofiniteness assumptions. We certainly believe that this result is true in general as long as $\mathbf{A}_n(M)$ is finite dimensional. A proof of this result without using the fusion matrices will be important and useful.

Note that our $\mathbf{A}_n(M)$ is different from $A_n(M)$ defined in [HY] where $A_n(M) = M/O_n(M)$ and $O_n(M)$ also contains $(L(-1) + L(0))M$. From the connection between intertwining operators and $\mathbf{A}_n(M)$ discussed below it seems that $(L(-1) + L(0))M$ should not be a subspace of $\mathbf{O}_n(M)$ in our consideration.

One of the important motivations for constructing $A_n(V)$ -bimodule $\mathbf{A}_n(M)$ is to study the extension of rational vertex operator algebras. It is a well known conjecture that if V is a rational vertex operator algebra then any extension U of V is also rational. It is expected that the $\mathbf{A}_n(M)$ -theory will play roles in proving this conjecture.

There are associative algebras $A_{g,n}(V)$ associated to an automorphism g of V of finite order and $n \in \frac{1}{o(g)}\mathbb{Z}_+$ [DLM4], [MT]. One could construct $A_{g,n}(V)$ -bimodule $A_{g,n}(M)$ for a g -twisted V -module M following the ideals of this paper.

The paper is organized as follows. We review the construction of associative algebras $A_n(V)$ and relevant results from [DLM3] in Section 2. The construction of $\mathbf{A}_n(M)$ is given in Section 3. We also show how the identity map on M induces an $A_n(V)$ -bimodule epimorphism from $\mathbf{A}_n(M)$ to $\mathbf{A}_{n-1}(M)$. Section 4 is devoted to the study of relation between $\mathbf{A}_n(M)$ and intertwining operators. As in [FZ] and [L2] we argue how the map from $\mathcal{I}_{M^i M^j}^{M^k}$ to $\text{Hom}_{A_n(V)}(\mathbf{A}_n(M^i) \otimes_{A_n(V)} M^j(s), M^k(t))$ by sending $I \in \mathcal{I}_{M^i M^j}^{M^k}$ to $I_{t,s}$ which maps $M^j(s)$ to $M^k(t)$ induces a bijection from $\mathcal{I}_{M^i M^j}^{M^k}$ to

$$\text{Hom}_{A_n(V)}(\mathbf{A}_n(M^i) \otimes_{A_n(V)} M^j(s), M^k(t))$$

if V is rational. Various properties of $\mathbf{A}_n(M)$ are discussed. In Section 5 we investigate the relation between $\mathbf{A}_n(M)^*$ and $\mathbf{A}_n(M')$.

2 $A_n(V)$ construction

This section is a review of the associative algebra $A_n(V)$ and related results from [DLM3]. Also see [Z] .

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra [B], [FLM], [LL]. We first recall different notions of modules from [DLM1, FLM, Z]. A *weak V-module* M is a vector space equipped with a linear map

$$\begin{aligned} Y_M(\cdot, z) : V &\rightarrow (\text{End} M)[[z, z^{-1}]] \\ v &\mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End} M) \end{aligned}$$

which satisfies the following conditions for $u \in V, v \in V, w \in M$ and $n \in \mathbb{Z}$,

$$\begin{aligned} u_n w &= 0 \text{ for } n \gg 0; \\ Y_M(\mathbf{1}, z) &= \text{id}_M; \\ z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) &- z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1) \\ &= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2). \end{aligned}$$

An (*ordinary*) V -module is a weak V -module M which carries a \mathbb{C} -grading induced by the spectrum of $L(0)$ where $L(0)$ is a component operator of

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

That is, $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. Moreover one requires that M_λ is finite dimensional and for fixed λ , $M_{n+\lambda} = 0$ for all small enough integers n .

An *admissible* V -module is a weak V -module M which carries a \mathbb{Z}_+ -grading $M = \bigoplus_{n \in \mathbb{Z}_+} M(n)$ that satisfies the following

$$v_m M(n) \subset M(n + \text{wt} v - m - 1)$$

for homogeneous $v \in V$. It is easy to show that any *ordinary* module is *admissible*.

For an *ordinary* V -module $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$, the contragredient module M' is defined in [FHL] as follows:

$$M' = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^*,$$

where M_λ^* is the dual space of M_λ . The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$\langle Y_{M'}(a, z)f, w \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})w \rangle,$$

where $\langle f, w \rangle = f(w)$ for $f \in M', w \in M$ is the natural pairing $M' \times M \rightarrow \mathbb{C}$.

V is called *rational* if every admissible V -module is completely reducible. It is proved in [DLM2] that if V is rational then there are only finitely many irreducible admissible V -modules up to isomorphism and each irreducible admissible V -module is ordinary. Let M^0, \dots, M^p be the irreducible modules up to isomorphism with $M^0 = V$. Then there exist $\lambda_i \in \mathbb{C}$ for $i = 0, \dots, p$ such that

$$M^i = \bigoplus_{n=0}^{\infty} M_{\lambda_i+n}^i$$

with $M_{\lambda_i}^i \neq 0$ where $L(0)|_{M_{\lambda_i+n}^i} = \lambda_i + n$ and $n \in \mathbb{Z}_+$. λ_i is called the *conformal weight* of M^i . We denote $M^i(n) = M_{\lambda_i+n}^i$. Moreover, λ_i and the central charge c are rational numbers (see [DLM5]). Let $(M^i)' = M^{i^*}$ for some unique $i^* \in \{0, \dots, p\}$.

V is called *C_2 -cofinite* if $\dim V/C_2(V) < \infty$ where $C_2(V) = \langle u_{-2}v | u, v \in V \rangle [\mathbb{Z}]$.

We now define algebra $A_n(V)$ for nonnegative integer n . Let $O_n(V)$ be the linear span of $u \circ_n v$ and $L(-1)u + L(0)u$ where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_n v = \text{Res}_z Y(u, z) v \frac{(1+z)^{\text{wt}u+n}}{z^{2n+2}}.$$

Also define second product $*_n$ on V for u and v as above:

$$\begin{aligned} u *_n v &= \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_z Y(u, z) \frac{(1+z)^{\text{wt}u+n}}{z^{n+m+1}} v \\ &= \sum_{m=0}^n \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\text{wt}u+n}{i} u_{i-m-n-1} v. \end{aligned}$$

Extend linearly to obtain a bilinear product on V . Set $A_n(V) = V/O_n(V)$.

The following theorem summarizes the main results of [DLM3].

Theorem 2.1. *Let V be a vertex operator algebra and n a nonnegative integer. Then*

- (1) $A_n(V)$ is an associative algebra whose product is induced by $*_n$.
- (2) The identity map on V induces an algebra epimorphism from $A_n(V)$ to $A_{n-1}(V)$.
- (3) Let W be a weak module and set

$$\Omega_n(W) = \{w \in W | u_m w = 0, u \in V, m > \text{wt}u - 1 + n\}.$$

Then $\Omega_n(W)$ is an $A_n(V)$ -module such that $v + O_n(V)$ acts as $o(v)$ where $o(v) = v_{\text{wt}v-1}$ for homogeneous v and extend linearly to entire V .

(4) Let $M = \bigoplus_{m=0}^{\infty} M(m)$ be an admissible V -module. Then each $M(m)$ for $m \leq n$ is an $A_n(V)$ -submodule of $\Omega_n(W)$. Furthermore, M is irreducible if and only if each $M(n)$ is an irreducible $A_n(V)$ -module for all n .

(5) V is rational if and only if $A_n(V)$ are finite dimensional semisimple algebras for all $n \geq 0$. In this case

$$A_n(V) = \bigoplus_{i=0}^p \bigoplus_{j=0}^n \text{End} M^i(j) = \bigoplus_{i=0}^p \bigoplus_{j=0}^n M^i(j) \otimes_{\mathbb{C}} M^{i^*}(j)$$

where M^i for $i = 0, \dots, p$ are the irreducible V -modules. Moreover, $O_{s-1}(V)/O_s(V)$ and $A_{s-1}(V)$ are two sided ideals of $A_n(V)$ for $s = 1, \dots, n$, and

$$O_{s-1}(V)/O_s(V) = \bigoplus_{i=0}^p \text{End} M^i(s).$$

(6) The map $M \mapsto M(n)$ gives one to one correspondence between irreducible admissible V -modules and the irreducible $A_n(V)$ -modules which are not $A_{n-1}(V)$ -modules.

The most parts of Theorem 2.1 are clear. We give a few words on Theorem 2.1 (5). By (1) the identity map on V induces an algebra epimorphism from $A_n(V)$ to $A_{n-1}(V)$ with kernel $O_{n-1}(V)/O_n(V)$. Since $A_n(V)$ is semisimple we conclude that both $O_{n-1}(V)/O_n(V)$ and $A_{n-1}(V)$ are the ideals of $A_n(V)$. It is now clear that

$$O_n(V)/O_{n-1}(V) = \bigoplus_{i=0}^p \text{End} M^i(n).$$

The decomposition of $O_s(V)/O_{s-1}(V)$ for arbitrary s follows immediately.

3 $A_n(V)$ -bimodules $\mathbf{A}_n(M)$

Let V be a vertex operator algebra and M be an admissible V -module. Motivated by the $A(V)$ -bimodule $A(M)$ from [Z] we define and study the $A_n(V)$ -bimodule $\mathbf{A}_n(M)$ for any nonnegative integer n . The construction of $\mathbf{A}_n(M)$ is largely influenced by the construction of $A_n(V)$ [DLM3] and the intertwining operators [FHL]. See [HY] for a different treatment.

Let $\mathbf{O}_n(M)$ be the linear span of $u \circ_n w$ where for homogeneous $u \in V$ and $w \in M$,

$$u \circ_n w = \text{Res}_z Y(u, z) w \frac{(1+z)^{\text{wt} u + n}}{z^{2n+2}}.$$

Also define a left bilinear product $*_n$ for $u \in V$ and $w \in M$:

$$\begin{aligned} u *_n w &= \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_z Y(u, z) \frac{(1+z)^{\text{wt} u + n}}{z^{n+m+1}} w \\ &= \sum_{m=0}^n \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\text{wt} u + n}{i} u_{i-m-n-1} w, \end{aligned} \quad (3.1)$$

and a right bilinear product

$$\begin{aligned} w *_n u &= \sum_{m=0}^n (-1)^n \binom{m+n}{n} \text{Res}_z Y(u, z) \frac{(1+z)^{\text{wt} u + m - 1}}{z^{n+m+1}} w \\ &= \sum_{m=0}^n \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\text{wt} u + m - 1}{i} u_{i-m-n-1} w. \end{aligned} \quad (3.2)$$

Set $\mathbf{A}_n(M) = M/\mathbf{O}_n(M)$. In the case $n = 0$ the $\mathbf{A}_0(M)$ is exactly the $A(V)$ -bimodule $A(M)$ studied in [FZ]

Remark 3.1. We have already mentioned that our $\mathbf{A}_n(M)$ is different from $A_n(M)$ defined in [HY] where $A_n(M) = M/O_n(M)$ and $O_n(M)$ also contains $(L(-1) + L(0))M$.

Remark 3.2. In the case $n = 0$ it follows immediately from the definitions that $\mathbf{A}_0(V) = A_0(V) = A(V)$ [Z]. But it is not clear if this is true in general. It will be established later that $\mathbf{A}_n(V) = A_n(V)$ for all n if V is rational.

Lemma 3.3. (1) Assume that $u \in V$ is homogeneous, $w \in M$ and $m \geq k \geq 0$. Then

$$\text{Res}_z Y(u, z) w \frac{(1+z)^{\text{wt}u+n+k}}{z^{2n+2+m}} \in \mathbf{O}_n(M).$$

(2) For homogeneous $u \in V$ and $w \in M$, $u *_n w - w *_n u - \text{Res}_z Y(u, z) w (1+z)^{\text{wt}u-1} \in \mathbf{O}_n(M)$.

Proof: The proof of (i) is similar to that of Lemma 2.1.2 of [Z]. (ii) follows from the definitions. \square

Lemma 3.4. We have the following containments:

- (1) $O_n(V) *_n M \subset \mathbf{O}_n(M)$, $M *_n O_n(V) \subset \mathbf{O}_n(M)$,
- (2) $V *_n \mathbf{O}_n(M) \subset \mathbf{O}_n(M)$, $\mathbf{O}_n(M) *_n V \subset \mathbf{O}_n(M)$.

Proof: The argument of containments $(L(-1) + L(0))V *_n M \subset \mathbf{O}_n(M)$, $M *_n (L(-1) + L(0))V \subset \mathbf{O}_n(M)$ is the similar to that of $(L(-1) + L(0))V *_n V \subset O_n(V)$, $V *_n (L(-1) + L(0))V \subset O_n(V)$ presented in the proof of Lemma 2.2 of [DLM3] using Lemma 2.1 (2). It remains to prove that $(u \circ_n v) *_n w$, $w *_n (u \circ_n v)$, $u *_n (v \circ_n w)$, $(u \circ_n w) *_n v \in \mathbf{O}_n(M)$ for $u, v \in V$ and $w \in M$. Since the proofs are similar we only prove $(u \circ_n v) *_n w \in \mathbf{O}_n(M)$.

We have

$$\begin{aligned} (u \circ_n v) *_n w &= \sum_{i \geq 0} \binom{\text{wt}u + n}{i} (u_{-2n-2+i} v) *_n w \\ &= \sum_{i \geq 0} \sum_{m=0}^n (-1)^m \binom{m+n}{n} \binom{\text{wt}u + n}{i} \text{Res}_{z_2} Y(u_{-2n-2+i} v, z_2) w \frac{(1+z_2)^{\text{wt}u+\text{wt}v+3n+1-i}}{z_2^{1+m+n}} \\ &= \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_{z_2} \text{Res}_{z_1-z_2} Y(Y(u, z_1-z_2)v, z_2) w \\ &\quad \cdot \frac{(1+z_1)^{\text{wt}u+n}(1+z_2)^{\text{wt}v+2n+1}}{(z_1-z_2)^{2n+2} z_2^{1+m+n}} \\ &= \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_{z_1} \text{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\text{wt}u+n}(1+z_2)^{\text{wt}v+2n+1}}{(z_1-z_2)^{2n+2} z_2^{1+m+n}} \\ &\quad - \sum_{m=0}^n (-1)^m \binom{m+n}{n} \text{Res}_{z_2} \text{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\text{wt}u+n}(1+z_2)^{\text{wt}v+2n+1}}{(z_1-z_2)^{2n+2} z_2^{1+m+n}} \\ &= \sum_{m=0}^n \sum_{i \geq 0} (-1)^m \binom{m+n}{n} \binom{-2n-2}{i} (-1)^i \\ &\quad \text{Res}_{z_1} \text{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\text{wt}u+n}(1+z_2)^{\text{wt}v+2n+1}}{z_1^{2n+2+i} z_2^{1+m+n-i}} \\ &\quad - \sum_{m=0}^n \sum_{i \geq 0} (-1)^m \binom{m+n}{n} \binom{-2n-2}{i} (-1)^i \\ &\quad \text{Res}_{z_2} \text{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\text{wt}u+n}(1+z_2)^{\text{wt}v+2n+1}}{z_1^{-i} z_2^{3n+3+m+i}}. \end{aligned}$$

From Lemma 3.3 we know that both

$$\text{Res}_{z_1} \text{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\text{wt}u+n} (1+z_2)^{\text{wt}v+2n+1}}{z_1^{2n+2+i} z_2^{1+m+n-i}}$$

and

$$\text{Res}_{z_2} \text{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\text{wt}u+n} (1+z_2)^{\text{wt}v+2n+1}}{z_1^{-i} z_2^{3n+3+m+i}}$$

lie in $\mathbf{O}_n(M)$. The proof is complete. \square

Recall from [DLM3] that the linear map

$$\phi : v \mapsto e^{L(1)} (-1)^{L(0)} v$$

induces an anti-isomorphism $A_n(V)$ to itself. We can now establish the following:

Theorem 3.5. *Let M be an admissible V -module and $n \geq 0$.*

- (1) *The $\mathbf{A}_n(M)$ is an $A_n(V)$ -bimodule such that the left and right actions of $A_n(V)$ on $\mathbf{A}_n(M)$ induced from (3.1) and (3.2), respectively.*
- (2) *The identity map on M induces an $A_n(V)$ -bimodule epimorphism from $\mathbf{A}_n(M)$ to $\mathbf{A}_{n-1}(M)$ if $n \geq 1$.*
- (3) *The map*

$$\phi : w \mapsto e^{L(1)} e^{\pi i L(0)} w$$

induces a linear isomorphism from $\mathbf{A}_n(M)$ to itself such that

$$\phi(u *_n w) = \phi(w) *_n \phi(u), \phi(w *_n u) = \phi(u) *_n \phi(w)$$

for $u \in V$ and $w \in M$.

- (4) *If V is rational then both $\mathbf{O}_{s-1}(M)/\mathbf{O}_s(M)$ and $\mathbf{A}_{s-1}(M)$ are the $A_n(V)$ -bimodules for $s = 1, \dots, n$ and*

$$\mathbf{A}_n(M) = \mathbf{A}_0(M) \bigoplus \bigoplus_{s=1}^n \mathbf{O}_{s-1}(M)/\mathbf{O}_s(M).$$

Proof: (1) By Lemma 3.4 it is good enough to prove the following relations in $\mathbf{A}_n(M)$ for $u, v \in V$ and $w \in M$:

$$\begin{aligned} (u *_n w) *_n v &= u *_n (w *_n v), \mathbf{1} *_n w = w *_n \mathbf{1} = w \\ (u *_n v) *_n w &= u *_n (v *_n w), w *_n (u *_n v) = (w *_n u) *_n v. \end{aligned}$$

Again the proofs are similar for these relations. We give a detail proof for $(u *_n w) *_n v = u *_n (w *_n v)$.

Using the definition we have

$$\begin{aligned}
u *_{\mathbf{n}} (w *_{\mathbf{n}} v) - (u *_{\mathbf{n}} w) *_{\mathbf{n}} v &= \sum_{m_1, m_2=0}^n (-1)^{m_1+n} \binom{m_1+n}{n} \binom{m_2+n}{n} \\
&\cdot \text{Res}_{z_1} \text{Res}_{z_2} Y(u, z_1) Y(v, z_2) w \frac{(1+z_1)^{\text{wt } u+n} (1+z_2)^{\text{wt } v+m_2-1}}{z_1^{n+m_1+1} z_2^{n+m_2+1}} \\
&- \sum_{m_1, m_2=0}^n (-1)^{m_1+n} \binom{m_1+n}{n} \binom{m_2+n}{n} \\
&\cdot \text{Res}_{z_2} \text{Res}_{z_1} Y(v, z_2) Y(u, z_1) w \frac{(1+z_1)^{\text{wt } u+n} (1+z_2)^{\text{wt } v+m_2-1}}{z_1^{n+m_1+1} z_2^{n+m_2+1}} \\
&= \sum_{m_1, m_2=0}^n (-1)^{m_1+n} \binom{m_1+n}{n} \binom{m_2+n}{n} \\
&\cdot \text{Res}_{z_2} \text{Res}_{z_1-z_2} Y(Y(u, z_1-z_2)v, z_2) w \frac{(1+z_1)^{\text{wt } u+n} (1+z_2)^{\text{wt } v+m_2-1}}{z_1^{n+m_1+1} z_2^{n+m_2+1}} \\
&= \sum_{m_1, m_2=0}^n \sum_{i, j \geq 0} (-1)^{m_1+n} \binom{m_1+n}{n} \binom{m_2+n}{n} \binom{\text{wt } u+n}{i} \binom{-n-m_1-1}{j} \\
&\cdot \text{Res}_{z_2} \text{Res}_{z_1-z_2} Y(Y(u, z_1-z_2)v, z_2) w \frac{(1+z_2)^{\text{wt } u+\text{wt } v+n+m_2-1-i} (z_1-z_2)^{i+j}}{z_2^{2n+m_1+m_2+2+j}} \\
&= \sum_{m_1, m_2=0}^n \sum_{i, j \geq 0} (-1)^{m_1+n} \binom{m_1+n}{n} \binom{m_2+n}{n} \binom{\text{wt } u+n}{i} \binom{-n-m_1-1}{j} \\
&\cdot \text{Res}_{z_2} Y(u_{i+j}v, z_2) w \frac{(1+z_2)^{\text{wt } u+\text{wt } v+n+m_2-1-i}}{z_2^{2n+m_1+m_2+2+j}}.
\end{aligned}$$

Note that $\text{wt } u_{i+j}v = \text{wt } u + \text{wt } v - i - j - 1$ and $\text{wt } u + \text{wt } v + n + m_2 - 1 - i = \text{wt } u_{i+j}v + n + j + m_2$. It follows from Lemma 3.3 that $\text{Res}_{z_2} Y(u_{i+j}v, z_2) w \frac{(1+z_2)^{\text{wt } u+\text{wt } v+n+m_2-1-i}}{z_2^{2n+m_1+m_2+2+j}}$ lies in $\mathbf{O}_n(M)$, as desired.

(2) By Lemma 3.3, $\mathbf{O}_n(M) \subset \mathbf{O}_{n-1}(M)$. So it is enough to show that $u *_{\mathbf{n}} w \equiv u *_{n-1} w, w *_{\mathbf{n}} u \equiv w *_{n-1} u$ modulo $\mathbf{O}_{n-1}(M)$ for $u \in V$ and $w \in M$. The proof is similar to that of Proposition 2.4 of [DLM3].

(3) We first prove that $\phi(\mathbf{O}_n(M)) \subset \mathbf{O}_n(M)$. Recall the following conjugation formulas from [FHL]:

$$\begin{aligned}
z^{L(0)} Y(u, z_0) z^{-L(0)} &= Y(z^{L(0)} u, z z_0), \\
e^{zL(1)} Y(u, z_0) e^{-zL(1)} &= Y\left(e^{z(1-zz_0)L(1)} (1-zz_0)^{-2L(0)} u, \frac{z_0}{1-zz_0}\right)
\end{aligned}$$

on M for $u \in V$. Then for $u \in V$ and $w \in M$,

$$\begin{aligned}
\phi(u \circ_n w) &= e^{L(1)} e^{\pi i L(0)} \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} u + n}}{z^{2n+2}} Y(u, z) w \\
&= \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} u + n}}{z^{2n+2}} e^{L(1)} Y((-1)^{L(0)} u, -z) e^{\pi i L(0)} w \\
&= \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} u + n}}{z^{2n+2}} Y \left(e^{(1+z)L(1)} (1+z)^{-2L(0)} (-1)^{L(0)} u, \frac{-z}{1+z} \right) e^{L(1)} e^{\pi i L(0)} w.
\end{aligned}$$

Making change of variable $z = -\frac{z_0}{1+z_0}$ and using the residue formula for the change of variable $[Z]$

$$\operatorname{Res}_z g(z) = \operatorname{Res}_{z_0} (g(f(z_0)) \frac{d}{dz_0} f(z_0))$$

(for $g(z) = \sum_{n \geq N} v_n z^{n+\alpha}$ and $f(z) = \sum_{n > 0} c_n z^n$ with $n \in \mathbb{Z}$, $v_n \in V$, $\alpha, c_n \in \mathbb{C}$) yields

$$\begin{aligned}
&\phi(u \circ_n w) \\
&= -\operatorname{Res}_{z_0} \frac{(1+z_0)^{-\operatorname{wt} u + n}}{z_0^{2n+2}} Y \left(e^{(1+z_0)^{-1} L(1)} (1+z_0)^{2L(0)} (-1)^{L(0)} u, z_0 \right) e^{L(1)} e^{\pi i L(0)} w \\
&= (-1)^{\operatorname{wt} u + 1} \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} u + n}}{z^{2n+2}} Y(e^{(1+z)^{-1} L(1)} u, z) e^{L(1)} e^{\pi i L(0)} w \\
&= (-1)^{\operatorname{wt} u + 1} \sum_{j=0}^{\infty} \frac{1}{j!} \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt} u + n - j}}{z^{2n+2}} Y(L(1)^j u, z) e^{L(1)} e^{\pi i L(0)} w \\
&= (-1)^{\operatorname{wt} u + 1} \sum_{j=0}^{\infty} \frac{1}{j!} \operatorname{Res}_z \frac{(1+z)^{\operatorname{wt}(L(1)^j u) + n}}{z^{2n+2}} Y(L(1)^j u, z) e^{L(1)} e^{\pi i L(0)} w
\end{aligned}$$

which lies in $\mathbf{O}_n(M)$ by definition.

Since the proof $\phi(u *_n w) = \phi(w) *_n \phi(u)$ and $\phi(w *_n u) = \phi(u) *_n \phi(w)$ are similar, we

give a proof of $\phi(w *_n u) = \phi(u) *_n \phi(w)$ only. We have

$$\begin{aligned}
\phi(w *_n u) &= \phi \left(\sum_{m=0}^n (-1)^n \binom{m+n}{n} \text{Res}_z Y(u, z) w \frac{(1+z)^{\text{wt } u+m-1}}{z^{n+m+1}} \right) \\
&= \sum_{m=0}^n (-1)^n \binom{m+n}{n} \text{Res}_z \frac{(1+z)^{\text{wt } u+m-1}}{z^{n+m+1}} e^{L(1)} Y((-1)^{L(0)} u, -z) e^{\pi i L(0)} w \\
&= \sum_{m=0}^n (-1)^n \binom{m+n}{n} \text{Res}_z \frac{(1+z)^{\text{wt } u+m-1}}{z^{n+m+1}} \\
&\quad \cdot Y(e^{(1+z)L(1)} (1+z)^{-2L(0)} (-1)^{L(0)} u, \frac{-z}{1+z}) e^{L(1)} e^{\pi i L(0)} w \\
&= \sum_{m=0}^n (-1)^{\text{wt } u+m} \binom{m+n}{n} \text{Res}_z \frac{(1+z)^{\text{wt } u+n}}{z^{n+m+1}} Y(e^{(1+z)^{-1}L(1)} u, z) e^{L(1)} e^{\pi i L(0)} w \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m=0}^n (-1)^{\text{wt } u+m} \binom{m+n}{n} \text{Res}_z \frac{(1+z)^{\text{wt } u+n-j}}{z^{n+m+1}} Y(L(1)^j u, z) e^{L(1)} e^{\pi i L(0)} w \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} (L(1)^j (-1)^{L(0)} u) *_n \phi(w) \\
&= \phi(u) *_n \phi(w).
\end{aligned}$$

The proof of (4) is similar to that of Theorem 2.1 (3). \square

We now take $M = V$. Theorems 2.1 and 3.5 give the following corollary.

Corollary 3.6. *Let $n \geq 0$. Then*

- (1) $\mathbf{A}_n(V)$ is an associative algebra with the product $*_n$ and the identity $\mathbf{1} + \mathbf{O}_n(V)$.
- (2) $\mathbf{O}_n(V)/\mathbf{O}_n(V)$ is a two sided ideal of $\mathbf{A}_n(V)$.
- (3) If V is rational then both $\mathbf{O}_{s-1}(V)/\mathbf{O}_s(V)$ and $\mathbf{A}_{s-1}(V)$ are two sided ideals of $\mathbf{A}_n(V)$ for $s = 1, \dots, n$ and

$$\mathbf{A}_n(V) = \mathbf{A}_0(V) \bigoplus \bigoplus_{s=1}^n \mathbf{O}_{s-1}(V)/\mathbf{O}_s(V).$$

We remark that $\omega + \mathbf{O}_n(V)$ does not lie in the center of $\mathbf{A}_n(V)$ as $\omega *_n u - u *_n \omega = (L(-1) + L(0))u$ for $u \in V$.

4 $\mathbf{A}_n(M)$ and intertwining operators

In this section we discuss how $\mathbf{A}_n(M)$ is related to the intertwining operators and fusion rules.

We first review the intertwining operators and tensor product of modules from [FHL]. Let (W^i, Y) be weak V -modules for $i = 1, 2, 3$. An intertwining operator of type $\begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$

is a linear map

$$\begin{aligned} W^1 &\rightarrow (\text{Hom}(W^2, W^3))\{z\} \\ u &\mapsto I(u, z) = \sum_{n \in \mathbb{C}} u_n z^{-n-1} \end{aligned}$$

satisfying the following conditions:

- (i) For any fixed $n \in \mathbb{C}$, $u \in W^1$, $v \in W^2$, $u_{n+k}v = 0$ for sufficiently large integer k ;
- (ii) $I(L(-1)u, z) = \frac{d}{dz}I(u, z)$ for $u \in W^1$;
- (iii) The Jacobi identity holds:

$$\begin{aligned} & z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) I(v, z_2) w - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) I(v, z_2) Y(u, z_1) w \\ &= z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y(u, z_0) v, z_2) w \end{aligned}$$

for any $u \in V$, $v \in W^1$ and $w \in W^2$.

Denote by $\mathcal{I} \left(\begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix} \right)$ the vector space of all intertwining operators of this type.

We call

$$N_{W^1 W^2}^{W^3} = \dim \mathcal{I} \left(\begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix} \right)$$

the fusion rules.

Now let $W^i = \bigoplus_{n=0}^{\infty} W^i(n)$ ($i = 1, 2, 3$) be three admissible V -modules such that $L(0)|_{W^i(n)} = (n + h_i)\text{id}$ for some constant $h_i \in \mathbb{C}$ for $i = 1, 2, 3$ with $W^i(0) \neq 0$. We write $\deg w = n$ for $w \in W^i(n)$. Then an intertwining operator $I \in \mathcal{I} \left(\begin{smallmatrix} W^3 \\ W^1 W^2 \end{smallmatrix} \right)$ can be written as

$$I(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} z^{-h_1-h_2+h_3}$$

for $v \in W^1$ (cf. Proposition 1.5.1 in [FZ]). It is clear from the definition that for homogeneous $u \in W^1$, $m, n \in \mathbb{Z}$,

$$u(n)W^2(m) \subseteq W^3(\deg u + m - n - 1).$$

We now can define the tensor product of two admissible V -modules. The tensor product for the admissible modules M, N is an admissible module $M \boxtimes N$ together with an intertwining operator $F \in \mathcal{I} \left(\begin{smallmatrix} M \boxtimes N \\ M N \end{smallmatrix} \right)$ satisfying the following universal mapping property: For any admissible V -module W and any intertwining operator $I \in \mathcal{I} \left(\begin{smallmatrix} W \\ M N \end{smallmatrix} \right)$,

there exists a unique V -homomorphism ψ from $M \boxtimes N$ to W such that $I = \psi \circ F$. Note that the definition of the tensor product does not guarantee the existence of tensor product.

From now on we assume that V is rational and C_2 -cofinite. Let M^0, \dots, M^p be the irreducible V -modules as before. For short we set $N_{ij}^k = N_{M^i M^j}^{M^k}$ for $i, j, k \in \{0, \dots, p\}$. Then N_{ij}^k are finite (cf. [ABD]). We also set $\mathcal{I}_{ij}^k = \mathcal{I} \begin{pmatrix} W^k \\ W^i W^j \end{pmatrix}$. Let $I_{ij}^{k,s}$ for $s = 1, \dots, N_{ij}^k$ be a basis of \mathcal{I}_{ij}^k . The following result is well known:

Theorem 4.1. *If V is rational then the tensor product $M^i \boxtimes M^j$ exists for any i, j . In fact,*

$$M^i \boxtimes M^j = \bigoplus_{k=0}^p \bigoplus_{s=1}^{N_{ij}^k} M^{k,s}$$

where $M^{k,s} \cong M^k$ as V -module and $F = \sum_{k=0}^p \sum_{s=1}^{N_{ij}^k} I_{ij}^{k,s}$ such that $I_{ij}^{k,s}(u, z)M^j \subset M^{k,s}\{z\}$.

The associativity of the tensor product is established in [H] with some additional assumptions:

Theorem 4.2. *If V is rational, C_2 -cofinite and self-dual, then the tensor product of V -modules is associative.*

We now turn our attention to the connection between intertwining operators and $\mathbf{A}_n(M)$. Let $I \in \mathcal{I}_{ij}^k$. Then for $w \in M^i$

$$I(w, z) = \sum_{m \in \mathbb{Z}} w(m) z^{-m-1} z^{-\lambda_i - \lambda_j + \lambda_k} \quad (4.3)$$

and $w(\deg w - 1 - t + s)M^j(s) \subset M^k(t)$ for all s, t . For short we set

$$o_{t,s}^I(w) = w(\deg w - 1 - t + s)$$

for homogeneous w and extend it linearly to entire M^i . The following result is a generalization of Lemma 1.5.2 of [FZ] and Theorem 3.2 of [DLM3].

Lemma 4.3. *The map $(w^i, w^j) \mapsto o_{t,s}^I(w^i)w^j$ for $w^i \in M^i, w^j \in M^j(s)$ for $s, t \leq n$ induces an $A_n(V)$ -module homomorphism $I_{t,s}$ from $\mathbf{A}_n(M^i) \otimes_{A_n(V)} M^j(s)$ to $M^k(t)$.*

Proof: As in [FZ] we need to verify that $o_{t,s}^I(u *_n w^i) = o(u)o_{t,s}^I(w^i)$, $o_{t,s}^I(w^i *_n u) = o_{t,s}^I(w^i)o(u)$ and $o_{t,s}^I(w) = 0$ on $M^j(s)$ with $s \leq n$ for $w^i \in M^i$ and $w \in \mathbf{O}_n(M^i)$. The proof is similar to that of Lemma 4.1 of [DJ]. \square

The proof of the following theorem is similar to that of Theorem 1.5.2 of [FZ] (see the proof of Theorem 2.11 of [L2]).

Theorem 4.4. *Assume that $M^j(s), M^k(t) \neq 0$ where $s, t \leq n$. Then the map from \mathcal{I}_{ij}^k to*

$$\text{Hom}_{A_n(V)}(\mathbf{A}_n(M^i) \otimes_{A_n(V)} M^j(s), M^k(t))$$

by sending I to $I_{t,s}$ is a linear isomorphism.

We need to review a well known result about finite dimensional simple algebra. Let A be a finite dimensional semisimple associative algebra and W an A -module. Then W^* is a right A -module such that $(fa)(w) = f(aw)$ for $a \in A$, $f \in W^*$ and $w \in W$.

Lemma 4.5. *If S, T are two simple A -modules then $S^* \otimes_A T = 0$ if S and T are inequivalent, and $S^* \otimes_A S = \mathbb{C}$.*

The following corollary tells us the bimodule structure of $\mathbf{A}_n(M^i)$ explicitly.

Corollary 4.6. *The $A_n(V)$ -bimodule $\mathbf{A}_n(M^i)$ has the decomposition*

$$\bigoplus_{j,k=0}^p \bigoplus_{s,t=0}^n N_{ij}^k M^k(t) \otimes_{\mathbb{C}} M^{j*}(s).$$

In particular,

$$\mathbf{A}_n(V) = \bigoplus_{j=0}^p \bigoplus_{s,t=0}^n M^j(t) \otimes_{\mathbb{C}} M^{j*}(s).$$

Proof: By Theorems 2.1, 3.5 and Lemma 4.5 we know that

$$\mathbf{A}_n(M^i) = \bigoplus_{j,k=0}^p \bigoplus_{s,t=0}^n a_{ij}^k(s, t) M^k(t) \otimes_{\mathbb{C}} M^{j*}(s)$$

as an $A_n(V)$ -bimodule for some nonnegative integers $a_{ij}^k(s, t)$. Theorem 4.4 then asserts that $a_{ij}^k(s, t) = N_{ij}^k$. \square

We remark that from the associativity of the tensor product (Theorem 4.2) one can easily prove that $A(M \boxtimes N) = A(M) \otimes_{A(V)} A(N)$ for any V -modules M, N . That is, A is functor from the V -module category to the $A(V)$ -bimodule category preserving the tensor product. However, the functor \mathbf{A}_n does not preserve the tensor product by Corollary 4.6.

The following corollary is a refinement of Theorem 3.5 (4).

Corollary 4.7. *Let $n \geq 1$. Then as a $A_n(V)$ -bimodule*

$$\mathbf{O}_{n-1}(M^i) / \mathbf{O}_n(M^i) = \bigoplus_{j,k=0}^p \bigoplus_{s=0}^n N_{ij}^k (M^k(n) \otimes M^{j*}(s) \oplus M^k(s) \otimes M^{j*}(n)).$$

We next investigate the relation among $\mathbf{A}_n(V)$ and $A_{t,s}(V)$ defined in [DJ]. The $A_{t,s}(V)$ is an $A_t(V) - A_s(V)$ -bimodule and the construction of $A_{t,s}(V)$ is motivated from the representation theory of vertex operator algebra. The $A_{t,s}(V)$ is defined as $V/O_{t,s}(V)$ where $O_{t,s}(V)$ is a subspace of V mainly containing $L(-1)u + (L(0) + s - t)u$ and

$$u \circ_s^t v = \text{Res}_z \frac{(1+z)^{wtu+s}}{z^{t+s+2}} Y(u, z)v$$

for $u, v \in V$. One can easily show that

$$\text{Res}_z \frac{(1+z)^{wtu+s+a}}{z^{t+s+2+b}} Y(u, z)v \in O_{t,s}(V)$$

for any nonnegative integer $a \leq b$ (cf. [DJ]). It is clear from the definition that $\mathbf{O}_n(V)$ is a subspace of $O_{t,s}(V)$ for any $s, t \leq n$. Using the proof of Theorem 3.5 we see that the identity map on V induces an epimorphism of $A_n(V)$ -bimodules from $\mathbf{A}_n(V)$ to $A_{t,s}(V)$ for $s, t \leq n$. Since V is rational here, $A_{t,s}(V)$ is a sub $A_n(V)$ -bimodule of $\mathbf{A}_n(V)$. In this case,

$$A_{t,s}(V) = \bigoplus_{i=0}^p \bigoplus_{l=0}^{\min\{t,s\}} M^i(t-l) \otimes M^{i*}(s-l)$$

by Theorem 4.16 of [DJ].

Finally, we interpret Corollary 4.6 in terms of intertwining operators. Recall a well known fact about associative algebra. If A is an associative algebra and M, N are A -modules then $\text{Hom}_{\mathbb{C}}(M, N)$ is an A -bimodule defined as follows: for $f \in \text{Hom}_{\mathbb{C}}(M, N)$, $w \in M$ and $a, b \in A$, $(afb)(w) = af(bw)$. If both M, N are finite dimensional then $\text{Hom}_{\mathbb{C}}(M, N)$ is isomorphic to $N \otimes_{\mathbb{C}} M^*$ as A -bimodule where M^* has a right A -module structure as mentioned before.

We now fix an intertwining operator $I \in \mathcal{I}_{ij}^k$. Recall the expression of $I(w, z)$ from equation (4.3). Set

$$O_{t,s}^I = \{o_{t,s}^I(w) | w \in M^i\} \subset \text{Hom}_{\mathbb{C}}(M^j(s), M^k(t)).$$

Lemma 4.8. *Let $s, t \leq n$. If $I \neq 0$ then $O_{t,s}^I = \text{Hom}_{\mathbb{C}}(M^j(s), M^k(t))$.*

Proof: Note that $o(a)o_{t,s}^I(w)o(b) = o_{t,s}^I(a *_n w *_n b)$ on $M^j(s)$ for $a, b \in V$ and $w \in M^i$ from the proof of Lemma 4.3. It follows immediately that $O_{t,s}^I$ is an $A_n(V)$ sub-bimodule of $\text{Hom}_{\mathbb{C}}(M^i(s), M^k(t))$. So it is sufficient to prove that $O_{t,s}^I \neq 0$ as long as $\text{Hom}_{\mathbb{C}}(M^i(s), M^k(t)) \neq 0$. Let $0 \neq u \in M^j(s)$. Using a result from [DM] and [L1] we know that

$$M^k = \langle w(m)u | w \in M^i, m \in \mathbb{Z} \rangle.$$

This implies that $M^k(t)$ is spanned by $o_{t,s}^I(w)u$ for $w \in M^i$. In particular, $O_{t,s}^I \neq 0$, as expected. \square

Recall that $I_{ij}^{k,m}$ for $m = 1, \dots, N_{ij}^k$ is a basis of \mathcal{I}_{ij}^k and $M^{k,m} \cong M^k$ are V -modules for $m = 1, \dots, N_{ij}^k$ as before. Then by Theorem 4.1 and Corollary 4.6 we have

Corollary 4.9. *As $A_n(V)$ -bimodule*

$$\mathbf{A}_n(M^i) = \bigoplus_{j,k=0}^p \bigoplus_{m=1}^{N_{ij}^k} \bigoplus_{s,t=0}^n O_{t,s}^{I_{ij}^{k,m}}.$$

5 The dual of $\mathbf{A}_n(M)$

In this section we study the dual of $\mathbf{A}_n(M)$ for a V -module M . First note that $\mathbf{A}_n(M)^*$ is also an $A_n(V)$ -bimodule such that for $a, b \in V$, $f \in \mathbf{A}_n(M)^*$ and $w \in M$, $(afb)(w) = f(b *_n w *_n a)$.

Lemma 5.1. *Let A be a finite dimensional semisimple associative algebra and M, N be the simple A -modules. Then the dual of A -module $M \otimes N^*$ is isomorphic to $N \otimes M^*$.*

Proof: Let $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_n\}$ be bases of M and N , respectively. Let $\{u_1^*, \dots, u_m^*\}$ and $\{v_1^*, \dots, v_n^*\}$ be the dual bases of M^* and N^* . Let $\{e_{ij} | i, j = 1, \dots, m\}$ be a basis of $\text{End} M \subset A$ such that $e_{ij}u_k = \delta_{j,k}u_i$ for all $1 \leq i, j, k \leq m$. Similarly, let $\{f_{ij} | i, j = 1, \dots, n\}$ be a basis of $\text{End} N \subset A$ such that $f_{ij}v_k = \delta_{j,k}v_i$ for $1 \leq i, j, k \leq n$. Note that $\{E_{ij} = u_i \otimes v_j^* | i = 1, \dots, m, j = 1, \dots, n\}$ is a basis of $M \otimes N^*$ and $\{F_{st} = v_s \otimes u_t^* | s = 1, \dots, n, t = 1, \dots, m\}$ is a basis of $N \otimes M^*$. Let $\{E_{ij}^* | i = 1, \dots, m, j = 1, \dots, n\}$ be the dual basis of $(M \otimes N^*)^*$.

We claim that $f_{st}E_{ij}^* = \delta_{t,j}E_{is}^*$ and $E_{ij}^*e_{ab} = \delta_{ia}E_{bj}^*$. Since the proofs of two relations are similar we only give the detail for the first relation. From the definition we know that

$$(f_{st}E_{ij}^*)(E_{ab}) = E_{ij}^*(E_{ab}f_{st}) = \delta_{b,s}E_{ij}^*(E_{at}) = \delta_{b,s}\delta_{a,i}\delta_{t,j}$$

for any a, b . That is, $f_{st}E_{ij}^* = \delta_{t,j}E_{is}^*$.

It is clear that the linear map from $(M \otimes N^*)^*$ to $N \otimes M^*$ by sending E_{ij}^* to F_{ji} is an A -bimodule isomorphism. \square

From Lemma 5.1 we see that there is a natural paring

$$(\cdot, \cdot) : M \otimes N^* \times N \otimes M^* \rightarrow \mathbb{C}$$

such that $(u \otimes f, v \otimes g) = f(v)g(u)$ for $u \in M, v \in N, f \in N^*, g \in M^*$. Moreover the following relation holds

$$(axb, y) = (x, bya)$$

for $x \in M \otimes N^*, y \in N \otimes M^*, a, b \in A$.

Proposition 5.2. *Let V be a rational, C_2 -cofinite vertex operator algebra. Assume that M^0, \dots, M^p are the irreducible V -modules such that $\lambda_i > 0$ if $i \neq 0$. Then for any $n \geq 0$ we have $\mathbf{A}_n(M^i)^*$ and $\mathbf{A}_n(M^{i*})$ are isomorphic $A_n(V)$ -bimodules. In particular, $A(M^i)^*$ and $A(M^{i*})$ are isomorphic $A(V)$ -bimodules.*

Proof: Recall that the fusion matrix $N(i) = (N_{ij}^k)_{j,k=0}^p$. By Lemma 5.5 of [DJX] we know that $N(i^*)$ and the transpose $N(i)^T$ of $N(i)$ are the same. That is, $N_{i^*j}^k = N_{ik}^j$ for all j, k . By Corollary 4.6, we have

$$\mathbf{A}_n(M^i) = \bigoplus_{j,k=0}^p \bigoplus_{s,t=0}^n N_{ij}^k M^k(t) \otimes_{\mathbb{C}} M^{j*}(s),$$

and

$$\mathbf{A}_n(M^{i*}) = \bigoplus_{j,k=0}^p \bigoplus_{s,t=0}^n N_{i^*k}^j M^j(s) \otimes_{\mathbb{C}} M^{k*}(t).$$

Lemma 5.1 then tells us that $\mathbf{A}_n(M^i)^*$ and $\mathbf{A}_n(M^{i*})$ are isomorphic $A_n(V)$ -bimodules. \square

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